Topological Data Analysis
MVA

Last name:
First name:

Duration: 20 minutes.

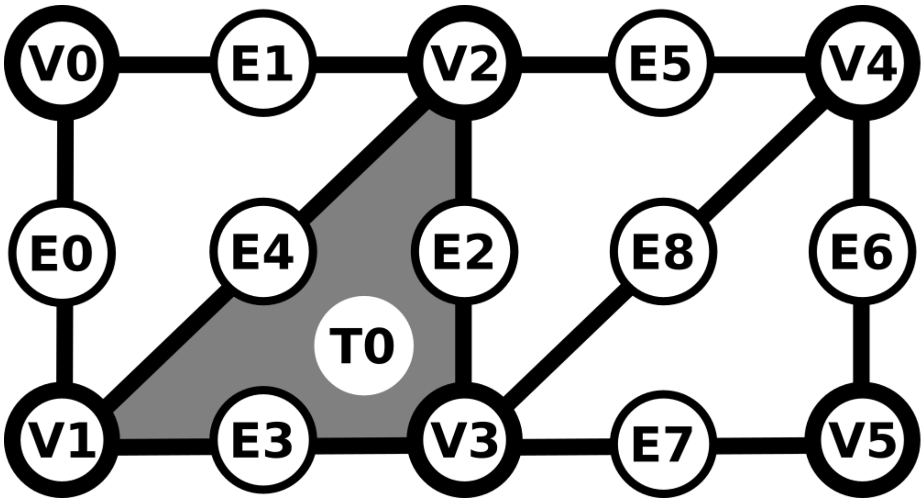

Figure 1: Simplicial complex $\mathcal{K}$, made of 6 vertices, 9 edges and 1 triangle.

Un-comment to mask the answers

Question 1: Let $\mathcal{K}$ be the simplicial complex shown in Figure 1. Give the Betti numbers of $\mathcal{K}$.

Question 2: Let $\mathcal{M}$ be a the triangulation of a 3-manifold, embedded in $\mathbb{R}^3$. For every interior $2-$simplex $\sigma_2 \in \mathcal{M}$, what is $\beta_0(Lk(\sigma_2))$ equal to?

Question 3: Let $\mathcal{K}$ be a filtered simplicial complex of dimension d: $\emptyset = \mathcal{K}^0 \subset \mathcal{K}^1 \subset \cdots \subset \mathcal{K}^m = \mathcal{K}$, s. t. $\mathcal{K}^{i+1} = \mathcal{K}^i \cup \sigma^{i+1}$ where σ^{i+1} is a simplex of $\mathcal{K}$. Recall that, for $i \in \{0, \cdots, m\}$ the simplex σ^i is said to be *positive* if it is contained in a $(k+1)$ cycle in K^i where $k+1 = \dim(\sigma^i)$, and *negative otherwise*. Recall also that the Betti numbers of $\mathcal{K}$ can be computed using the following algorithm.

Input: The filtration $\emptyset = \mathcal{K}^0 \subset \mathcal{K}^1 \subset \cdots \subset \mathcal{K}^m = \mathcal{K}$.
Output: The Betti numbers $\beta_0, \beta_1, \cdots, \beta_d$ of K.

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β_0 = β_1 = ··· = β_d = 0;
for i = 1 to m
  k = dim σ^i − 1;
  if σ^i is positive
    then β_{k+1} = β_{k+1} + 1;
    else β_k = β_k − 1;
  end if;
end for;
output (β_0, β_1, ··· , β_d);
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For $k = 0, \cdots, d$ let N_d be the number of simplices of dimension k in $\mathcal{K}$. Show the following equality (equivalent definitions of the Euler characteristic):

$$\sum_{k=0}^{d} (-1)^k N_k = \sum_{k=0}^{d} (-1)^k \beta_k.$$

(Hint: β_k can be expressed as a function of the number of positive simplices of dimension k and of the number of negative simplices of dimension $k+1$.)

Question 4: To compute the 1-dimensional persistent homology of a filtered simplicial complex, it is enough to know its 1-skeleton (i.e. the set of its vertices and edges):
a) True
b) False

Question 5: The persistence diagram of the distance function to the unit sphere in $\mathbb{R}^3$ is given by (using Gudhi notations $(dimension, (birth, death))$):
a) $[(0,(0.0,\infty)),(1,(0.0,1.0)),(2,(0.0,1.0))]$
b) $[(0,(0.0,\infty)),(2,(0.0,1.0))]$
c) $[(0,(0.0,\infty)),(1,(0.0,1.0))]$

Question 6: Let D_1 and D_2 be the two persistence diagrams defined by:

$$D_1 = [(1,(1.0,2.0)),(1,(2.0,4.0)),(1,(3.0,3.5))] \tag{1}$$

and

$$D_2 = [(1,(0.5,1.0)),(1,(1.8,4.2)),(1,(2.7,3.5))]. \tag{2}$$

The bottleneck distance between D_1 and D_2 is equal to:
a) 1.0
b) 0.3
c) 0.5

Question 7: Let $f : \mathcal{M} \to \mathbb{R}$ be a PL scalar function defined on a closed, orientable PL 2-manifold $\mathcal{M}$, admitting a *monkey* saddle on the vertex v_s. What are $\beta_0(Lk^-(v_s))$ and $\beta_0(Lk^+(v_s))$ equal to?

Question 8: Let $f : \mathcal{M} \to \mathbb{R}$ be a PL *Morse* function defined on a closed, orientable PL 2-manifold $\mathcal{M}$ of genus 5, such that f admits only 1 minimum and 2 maxima. What is the number of loops of the Reeb graph $\mathcal{R}(f)$ of f?

Question 9: Let $\hat{f} : \mathcal{M}^0 \rightarrow \mathbb{R}$ be the scalar field evaluated on the vertices of a PL 2-manifold $\mathcal{M}$ and represented in Figure 2. Construct a valid discrete gradient field for this example and draw it on Figure 2 (use different representations for vertex-edge and edge-triangle discrete vectors).

Question 10: Enumerate the critical vertices, edges and triangles for the previous question. Describe a perturbation procedure for the removal of the critical triangle.

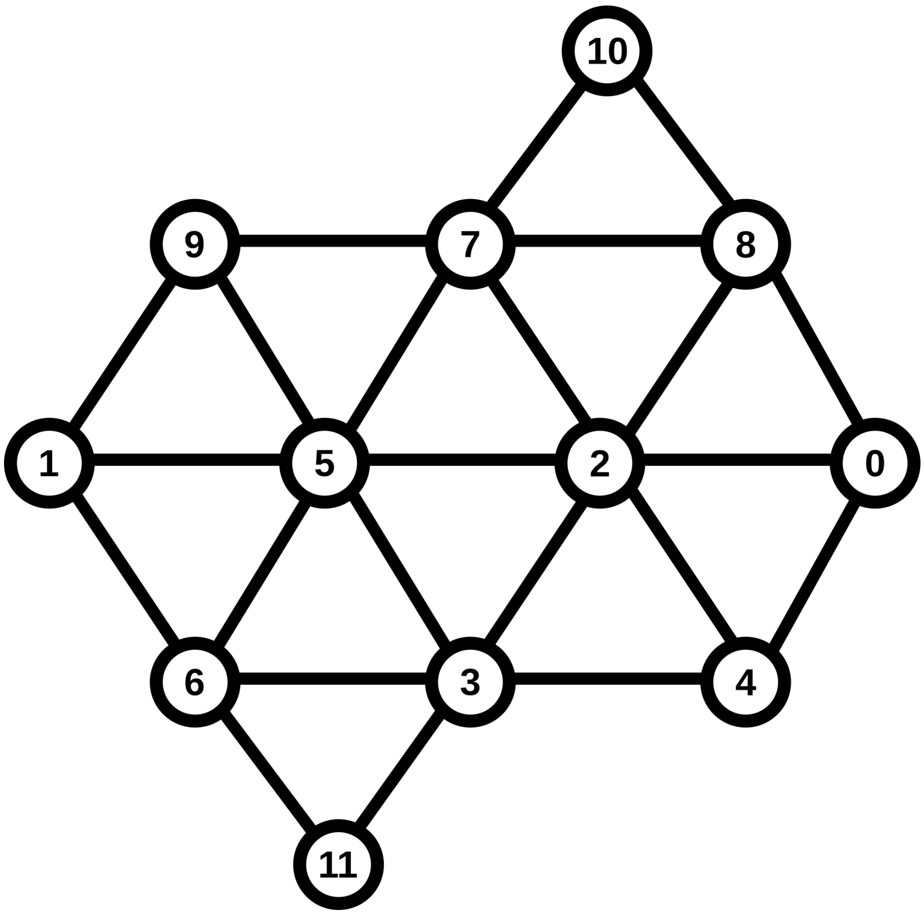

Figure 2: PL 2-manifold $\mathcal{M}$ made of 12 vertices, 23 edges and 12 triangles. The function $\hat{f} : \mathcal{M}^0 \to \mathbb{R}$ evaluated on the vertices of $\mathcal{M}$ is represented by a number in each vertex.